Journal of Engineering Physics and Thermophysics, Vol. 75, No. 2, 2002

PERFORMANCE CRITERIA FOR ANALYTICAL CALCULATION OF THE NONSTATIONARY TEMPERATURE FIELD IN THE ACTIVE ELEMENT OF AN ELECTROMAGNET

V. S. Loginov[a] and A. R. Dorokhov[b*]

UDC 621.039.534.54,621.364,634.3

To justify the accuracy of the engineering calculation of the nonstationary temperature field in an active element of finite size, we propose to introduce into practice the performance criteria of calculation.

In connection with the advent of new energy-conserving technologies which are associated with the use of thermal processes, it became necessary to justify the quality of calculation of nonstationary temperature fields in various fuel elements (power transformers and capacitors, electric machines and apparatuses, charged-particle accelerators). This performance criterion largely affects the following factors:

1) making constructive decisions in choosing weight and overall dimensions of the entire power installation in order to provide reliable admissible temperature conditions;

2) justification of the approximate or numerical method of thermal calculation of an electromagnet;

3) rational approach to solution of the inverse problems which are associated with determination of the energy-thermophysical properties of materials (heat release, coefficients of heat and mass exchange, and dielectric and other characteristics).

In these cases, it is important to have reliable information which can be obtained either from expensive experimental investigations or from calculations according to a known analytical solution of the classical or numerical problem of heat exchange.

At the present time, the development of computational engineering makes it possible to investigate more comprehensively analytical and numerical methods of the solution of problems, in particular, of heat conduction in solids of finite size in a wide range of variation of different parameters.

Many analytical solutions of the linear nonstationary problems of heat conduction exist [1–4] or can be solved analytically [5] or by other numerical-analytical methods [6–9]. However, because of the complexity of the analytical solutions [2], related to the convergence of series sums, these solutions have not yet received practical implementation. Moreover, in calculations, one can frequently obtain data contradicting the physical meaning of the problem.

The justification of other methods of solution [6–9] is based on a comparison of the results of numerical experiments to the data calculated by analytical expressions with rigorously and specifically chosen parameters of the initial problem. This approach was the only one possible in the absence of powerful facilities of computational engineering. At the present time, it is necessary to obtain reliable information from the analytical solution of the initial problem at the early stage of simulation of a thermal process in a specific active element. Of particular importance is verification of calculation results, which implies the substitution of the calculated values into the initial differential equations and boundary conditions of the investigated problem. Unfortunately, this stage has not received sufficient attention in the literature, but it is quite essential in investigations. Only after this stage is it recommended to perform the comparison to the experimental or other reliable data and to proceed to the simulation process itself in a wide range of variation of the parameters of the initial problem.

The objective of the present work is to justify the specific problem of the performance criteria for analytical calculation of the nonstationary temperature field in the active element.

---

* Deceased

[a]Tomsk Polytechnic University, Tomsk, Russia; [b]Kuzbas State Technical University, Kemerovo, Russia. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 75, No. 2, pp. 148–151, March–April, 2002. Original article submitted February 20, 2001; revision submitted August 22, 2001.

In the heat-conduction theory [1], the Fourier number is considered to be an important index of convergence of a one-dimensional series. For example, for an unbounded plate with $\mathrm{Fo} \geq 0.3$ one can restrict oneself to one first term of the series. When the initial problem is complicated, as will be shown below, it is evidently insufficient to have one value of the Fourier number for obtaining reliable data.

Let us consider, for example, the two-dimensional problem of nonstationary heat conduction with internal heat sources dependent on coordinates and time under asymmetric cooling conditions when the initial condition is zero. This problem is most frequently encountered in thermal calculations of electromagnetic devices. The system of equations of this problem has the form

$$\frac{\partial \theta}{\partial \,\mathrm{Fo}} = \frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Y^2} + \mathrm{Po}\,(X, Y, \mathrm{Fo}) \,, \quad 0 < X < 1 \,, \quad \mathrm{Fo} > 0 \,, \tag{1}$$

with the following boundary and initial conditions:

$$\left[\frac{\partial \theta}{\partial X} + \mathrm{Bi}_1 \theta\right]_{X=1} = 0 \,, \quad \left[\frac{\partial \theta}{\partial X} - \mathrm{Bi}_2 \theta\right]_{X=0} = 0 \,, \quad \left[\frac{\partial \theta}{\partial Y} + \mathrm{Bi}_3 \theta\right]_{Y=R} = 0 \,, \quad \left[\frac{\partial \theta}{\partial Y} - \mathrm{Bi}_4 \theta\right]_{Y=0} = 0 \,; \tag{2}$$

$$\theta\,(X, Y, 0) = 0 \,. \tag{3}$$

The solution of this system of equations according to [5] has the form

$$\theta\,(X, Y, \mathrm{Fo}) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{T_1\,(\mu_n, \gamma_m, \mathrm{Fo})\, K_1\,(\mu_n, X)\, K_2\,(\gamma_m, Y)}{K_{11}\,(\mu_n)\, K_{22}\,(\gamma_m)} \,. \tag{4}$$

Here μ_n and γ_m are the eigenvalues. They are found from the transcendental equations

$$\mathrm{ctan}\, \mu = \frac{\mu^2 - \mathrm{Bi}_1\, \mathrm{Bi}_2}{\mu\,(\mathrm{Bi}_1 + \mathrm{Bi}_2)} \,, \tag{5}$$

$$\mathrm{ctan}\, \gamma R = \frac{\gamma^2 - \mathrm{Bi}_3\, \mathrm{Bi}_4}{\gamma\,(\mathrm{Bi}_3 + \mathrm{Bi}_4)} \,, \tag{6}$$

$$K_1\,(\mu_n, X) = \mu_n \cos \mu_n X + \mathrm{Bi}_2 \sin \mu_n X \,, \quad K_2\,(\gamma_m, Y) = \gamma_m \cos \gamma_m Y + \mathrm{Bi}_4 \sin \gamma_m Y \,,$$

$$K_{11}\,(\mu_n) = \frac{1}{2}\left\{\mu_n^2 + \mathrm{Bi}_2^2 + (\mu_n^2 - \mathrm{Bi}_2^2)\, \frac{\sin 2\mu_n}{2\mu_n} + \mathrm{Bi}_2\,(1 - \cos 2\mu_n)\right\} ,$$

$$K_{22}\,(\gamma_m) = \frac{1}{2}\left\{(\gamma_m^2 + \mathrm{Bi}_4^2)\, R + (\gamma_m^2 - \mathrm{Bi}_4^2)\, \frac{\sin 2\gamma_m R}{2\gamma_m} + \mathrm{Bi}_4\,(1 - \cos 2\gamma_m R)\right\} . \tag{7}$$

$$T_1\,(\mu_n, \gamma_m, \mathrm{Fo}) = \int_0^1 \int_0^R \int_0^{\mathrm{Fo}} \mathrm{Po}\,(X', Y', \mathrm{Fo}')\, \exp\,[-\,(\mu_n^2 + \gamma_m^2)\,(\mathrm{Fo} - \mathrm{Fo}')]\, K_1(\mu_n, X')\, K_2\,(\gamma_m, Y')\, dX'\, dY'\, d\mathrm{Fo}' \,.$$

Let us consider the case of distribution of the specific loss in electric machines which frequently occurs in practice:

TABLE 1. Eigenvalues μ_n and γ_m Determined from Eqs. (5) and (6)

$n,\ m$	μ_n	γ_m	$n,\ m$	μ_n	γ_m
1	1.382665	0.30031	15	44.03678	5.90008
2	3.754395	0.63786	16	47.17475	6.31665
3	6.639568	1.00477	17	53.45196	6.73349
4	9.671262	1.38902	18	56.59106	7.15056
5	12.75381	1.78388	19	59.73043	7.56783
6	15.85891	2.18548	20	62.87002	7.98526
7	18.97581	2.59151	21	66.00979	8.40283
8	22.09960	3.00055	22	69.00979	8.82053
9	25.22778	3.41169	23	72.28983	9.23833
10	28.35890	3.82435	24	75.43038	9.65622
11	31.49209	4.2383	25	78.57037	10.0742
12	34.62679	4.65276	26	81.71077	10.4922
13	37.76264	5.06805	27	87.99187	10.91035

TABLE 2. Influence of the Number of Terms in Series (4)–(7) on the Temperature at the Point of the Active Element for Fo = 0.6, $N = 2$, $s = 0.5$, $D = -1/R^2 = -0.017778$, and Po(X, Y, Fo) = 28.616

θ	$\partial\theta/\mathrm{Fo}$	η_1	η_2	ζ	$i,\ j$	$k,\ p$
14.40	10.93	–16.13	–1.19	0.36	3; 4	20; 20
14.43	11.07	–16.22	–1.85	–0.52	3; 4	10; 10
14.42*		–15.54	–1.82	+0.18	3; 3	
14.70	11.25	–16.80	–1.74	–1.74	3; 4	5; 5
14.40	10.90	–16.12	–0.92	0.67	3; 4	30; 30
8.50	6.43	–16.24	–0.77	5.17	1; 1	
14.88	11.60	–28.46	–3.36	–14.80	1; 2	
15.78	12.42	–30.17	–4.27	–18.24	1; 3	5; 5
15.28	11.87	–29.21	–3.30	–15.76	1; 4	
14.80	11.25	–28.30	–1.79	–12.73	1; 5	

* $\eta_1 = \dfrac{\partial^2\theta}{\partial X^2}$, $\eta_2 = \dfrac{\partial^2\theta}{\partial Y^2}$, ξ = Po(X, Y, Fo) $- \dfrac{\partial^2\theta}{\partial \mathrm{Fo}} + \dfrac{\partial^2\theta}{\partial X^2} + \dfrac{\partial^2\theta}{\partial Y^2}$ is the discrepancy in the energy equation.

$$\mathrm{Po}\,(X, Y, \mathrm{Fo}) = \mathrm{Po}_0 W_1\,(X)\, W_2\,(Y) \exp\,(-\,s\,\mathrm{Fo})\,.$$

Here $W_1(X) = \exp\,(-\,NX)$ and $W_2(Y) = 1 + MY + DY^2$.

For this case the function (7) takes the specific form

$$T_1\,(\mu_n, \gamma_m, \mathrm{Fo}) = \mathrm{Po}_0 F_1\,(\mu_n)\, F_2\,(\gamma_m, R)\, F_3\,(\mu_n, \gamma_m, \mathrm{Fo})\,,$$

where

$$F_1\,(\mu_n) = \frac{\mu_n^2}{(\mu_n^2 + N^2)}\left\{\left[\left(1 - \frac{\mathrm{Bi}_2 N}{\mu_n^2}\right)\sin\mu_n - \frac{1}{\mu_n}\,(N + \mathrm{Bi}_2)\cos\mu_n\right]\exp\,(-\,N) + \frac{1}{\mu_n}\,(N + \mathrm{Bi}_2)\right\};$$

$$F_2\,(\gamma_m, R) = \left[W_2\,(R) - \frac{2D}{\gamma_m^2}\right]\left(\sin\gamma_m R - \frac{\mathrm{Bi}_4}{\gamma_m}\cos\gamma_m R\right) + \frac{1}{\gamma_m^2}\,(M + 2DR)\, K_2\,(\gamma_m R) + \frac{\mathrm{Bi}_4}{\gamma_m}\left(1 - \frac{2D}{\gamma_m^2}\right) - \frac{M}{\gamma_m}\,;$$

$$W_2\,(R) = 1 + MR + DR^2\,; \quad K_2\,(\gamma_m, R) = \gamma_m\cos\gamma_m R + \mathrm{Bi}_4\sin\gamma_m R\,;$$

TABLE 3. Evaluation of the Influence of the Initial Data on the Results of Calculations for Fo = 0.1, R = 7.5, Po_0 = 112, X_* = 0.5, Y_* = 0.5R, $N = s$ = 2.0, $D = -1.7778 \cdot 10^{-2}$, and Bi_i = 0.00001 (i = 1, 2, 3, 4). [According to Eqs. (5) and (6), These Biot Numbers Correspond to the Following Eigenvalues ($\mu_n = \gamma_m R$): 0.0001, 3.142, 6.283, 9.425, 12.566, ...; $k = p$ = 5]

Number of variant	Bi_i	n; m	ζ	θ	$\mu_1 = \gamma_1 R$	B
1	0.00001	2; 3	–41.1	5.87	0.0001	0.347
		4; 2	–29.65	5.59		$1.4 \cdot 10^{-4}$
		5; 5	–32.81	5.61		$1 \cdot 10^{-7}$
2	0.0001	4; 5	+93.7	–7.32	0.0001	$1 \cdot 10^{-4}$
3	0.0001	4; 5	–5.86	3.101	0.001	$1 \cdot 10^{-4}$
4	0.0001	4; 5	$2.9 \cdot 10^{6}$	$-3 \cdot 10^{5}$	0.00001	$1 \cdot 10^{-4}$
5	0.001	5; 5	$-4 \cdot 10^{3}$	380.2	0.0001	$1 \cdot 10^{-7}$
6	0.001	3; 4	–4.86	3.09	0.001	$1.6 \cdot 10^{-2}$
		4; 5	–5.87	3.10		$1.4 \cdot 10^{-4}$
		5; 5	–7.61	3.11		1.10^{-7}
7*	0.001	3; 3	0.06	11.20	0.001	1.10^{-2}
8**		4; 5	–3.31	8.31		1.10^{-4}
		5; 4				$1.2 \cdot 10^{-7}$

* Constant heat release with $D = N = M = s = 0$ and Po_0 = 112.

** Under heat release with Po_0 = 112, $D = -1.7778 \cdot 10^{-2}$, and $B = \exp\,[-(\mu_n^2 + \gamma_m^2 + s)\ \mathrm{Fo}]$.

$$F_3\,(\mathrm{Fo}) = \frac{1}{\mu_n^2 + \gamma_m^2 + s} \left\{ 1 - \exp \left[-\,(\mu_n^2 + \gamma_m^2 + s)\ \mathrm{Fo} \right] \right\} .$$

Practical implementation of the solution (4)–(7) is associated with application of a limited number of terms in the series (4). Therefore, the natural question arises: "How can this limitation affect the accuracy of the final results of calculations?" As an example, we consider a pressure plate of a turbogenerator [10]. The initial data for the calculations are as follows: Bi_1 = 0.8, Bi_2 = 1.6, Bi_3 = 0.4, Bi_4 = 1.2, R = 7.5, Po = 112.0, $N = M = s = 0$, and $D = -1/R^2$. From Eqs. (5) and (6) we find the eigenvalues. Their values are given in Table 1.

The next roots of the eigenvalues in the first approximation can be determined from the simple dependences

$$\mu_{n+1} = \mu_n + \pi\,, \quad \gamma_{m+1} = \gamma_m + \pi/R\,.$$

For these eigenvalues we calculate the temperature fields at different Fourier numbers and parameters of heat releases. Table 2 gives the calculated values (for the points X = 0.5 and Y = 0.25R) of the temperatures, the rate of temperature change with time, and the discrepancy of the energy equation. From this table it is seen that the reliable results on calculation of the thermal state of the active element under nonstationary conditions can be obtained only in the case where the minimum discrepancy in the energy equation occurs. Otherwise, one can obtain incorrect results of calculations. It is quite obvious that in the present case the convergence of the double series is nonuniform. We carried out the calculations on elucidating the influence of the accuracy of the initial data and determination of the final results of calculations. It has been established that the greatest calculation error is observed for the case where the cooling is absent (Table 3).

Therefore, at large Fourier numbers we can restrict ourselves to a comparatively small discrepancy of 10^{-4} in determining the eigenvalues, since the exact value of the temperature $\theta = 14.40$ differs little from its approximate value of 14.42 (Table 2). As far as the satisfaction of the boundary conditions is concerned, we can say the following. If in the calculations the minimum discrepancy in the energy equation was achieved and the boundary-value problem was posed and solved correctly, the boundary conditions are satisfied with a rather high accuracy. In our examples, the discrepancies in the boundary conditions were of the order from 2.10^{-5} to 1.10^{-4} depending on the accuracy of determination of the eigenvalues for the boundary-value problem.

As is seen from Table 3, neglect of the value of the discrepancy in the energy equation can lead to incorrect calculation results which are related to the reasonable accuracy of the initial data and to the accumulation of errors in the course of evaluations. Therefore, as the final calculation results we take the values corresponding to variant 6: $\zeta = -5.87$, $\theta = 3.10$ for $\mathrm{Bi}_i = 0.001$, and $\mu_1 = \gamma_1 R = 0.001$. Comparison with variants 7 and 8 shows that for a constant heat release the energy equation is fulfilled more exactly than in variant 8 and less exactly with a sharp change in the heat release (variant 6).

Thus, the performance criteria for calculating the nonstationary two-dimensional temperature field in the active element with a sharp change in the heat release with respect to the coordinates and time are the values for the minimum discrepancies in the energy equation, the boundary conditions, and the exponential factor that depends on the Fourier number. Along with the final results it is recommended to give the values of the minimum discrepancies in the energy equation and the boundary conditions in justifying any method of calculation (analytical, numerical, or approximate) of the problem of engineering thermal physics.

NOTATION

$\theta = [T(x, y, \tau) - T_{\text{cool}}]/T_{\text{sc}}$, dimensionless temperature, $T(x, y, \tau)$, T_{cool}, and T_{sc}, corresponding temperatures; $\mathrm{Po}(X, Y, \mathrm{Fo}) = q_v(x, y, \tau)b^2/(\lambda_x T_{\text{sc}})$, Pomerantsev number; N, D, M, and s, components of the heat release; $\mathrm{Bi}_{1,2} = \alpha_{1,2}b/\lambda_x$ and $\mathrm{Bi}_{3,4} = \alpha_{3,4}b/\sqrt{\lambda_x\lambda_y}$, Biot numbers; $X = x/b$, $Y = \dfrac{y}{b}\left(\dfrac{\lambda_x}{\lambda_y}\right)^{1/2}$, and $R = \dfrac{H}{b}\left(\dfrac{\lambda_x}{\lambda_y}\right)^{1/2}$, dimensionless coordinates; b and H, geometric dimensions; λ_x and λ_y, coefficients of thermal conductivity; α_i ($i = 1, 2, 3, 4$), coefficients of heat exchange; $\mathrm{Fo} = a\tau/b^2$, Fourier number; n and m, numbers of series terms; i and j, number of the series terms at which the minimum discrepancy in the energy equation is observed; k and p, limited number of the terms of every series used in calculating the specific variant of the problem under consideration. Subscripts: cool, coolant; sc, scale.

REFERENCES

1. A. V. Luikov, *Theory of Heat Conduction* [in Russian], Moscow (1967).
2. A. V. Luikov, *Heat and Mass Transfer. Handbook* [in Russian], Moscow (1971).
3. É. M. Kartashov, *Izv. Ross. Akad. Nauk, Énergetika*, No. 5, 3–34 (1999).
4. A. D. Polyanin, *Teor. Osn. Khim. Tekhnol.*, **34**, No. 6, 563–564 (1999).
5. N. S. Koshlyakov, É. B. Gliner, and M. M. Smirnov, *Partial Differential Equations of Mathematical Physics* [in Russian], Moscow (1979).
6. V. L. Rvachev, A. P. Slesarenko, and N. D. Sizova, *Inzh.-Fiz. Zh.*, **39**, No. 3, 526–531 (1980).
7. G. N. Dul'nev, Yu. L. Gur'ev, and S. G. Suslov, *Inzh.-Fiz. Zh.*, **30**, No. 3, 520–526 (1980).
8. G. M. Kobel'kov, *Zh. Vych. Mat. Mat. Fiz.*, **41**, No. 3, 407–419 (2001).
9. V. I. Mazhukin, D. A. Malafei, P. P. Matus, and A. A. Samarskii, *Zh. Vych. Mat. Mat. Fiz.*, **41**, No. 3, 407–419 (2001).
10. V. G. Dan'ko, *Élektrotekhnika*, No. 10, 11–13 (1970).